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STOCHASTIC AVERAGING OF STRONGLY NON-LINEAR OSCILLATORS UNDER COMBINED HARMONIC AND WHITE-NOISE EXCITATIONS

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A stochastic averaging procedure of strongly non-linear oscillators subject to external and (or) parametric excitations of both harmonic and white-noise forces is developed by using the so-called generalized harmonic functions. The procedure is applied to a Duffing oscillator with hardening stiffness under both external harmonic excitation and external and parametric excitations of white noises. The averaged Fokker–Planck–Kolmogrov equation is solved by using the path integration method. Based on the stationary joint probability density of amplitude and phase obtained by using the stochastic averaging and the path integration, the stochastic jump of the Duffing oscillator under combined harmonic and white-noise excitations and its bifurcation as the system parameters (frequency ratio, strength of the non-linearity, amplitude of harmonic excitation and intensity of white noise) change are examined for the first time.

1. INTRODUCTION

The stochastic averaging method is a powerful approximate technique for the prediction of response, decision of stability and estimation of reliability of linear or non-linear conservative oscillators subject to lightly linear and (or) non-linear dampings and external (additive) and (or) parametric (multiplicative) excitations of the wideband random processes. It has been extensively used in theoretical investigation and engineering application of random vibration. Comprehensive reviews attesting the success of the stochastic averaging method in random vibration have been written by Roberts and Spanos [2] and by Zhu [2, 3]. The success of the stochastic averaging method is mainly due to its two advantages: the equations of motion of a system are much simplified and the dimensions of the equation is often reduced while the essential behavior of the system is retained; the averaged response is a diffusive Markov process and the method of Fokker–Planck–Kolmogorov (FPK) equation can be applied.

The classical stochastic averaging method has two widely used versions. One is often called the standard stochastic averaging or Stratonovich stochastic averaging. This version of the stochastic averaging method was initially developed by Stratonovich [4] and later justified mathematically by Khasminskii [5]. It can be applied to multi-degree-of-freedom (m.d.o.f.) quasi-linear random systems, i.e., linear conservative oscillators subject to lightly linear and (or) non-linear dampings and weakly external and (or) parametric excitations of wideband random processes. The other version is usually called the stochastic averaging of energy envelope or quasi-conservative averaging. It was initially proposed by Khasminiskii [6] and Landa and Stratonovich [7]. Later, this version of the stochastic averaging equation was rederived by Zhu [8] and Zhu and Lin [9] based on a theorem due to Khasminskii [10], and possible change of the energy envelope due to the conservative part of the Wong–Zakai correction terms was taken into account. This version of the stochastic averaging method is applicable to single-degree-of-freedom (s.d.o.f.) non-linear conservative oscillators subject to lightly linear and (or) non-linear dampings and weakly external and (or) parametric excitations of Gaussian white noises. Later, it was extended to the case of non-white wideband random excitations by Roberts [11] and Cai [12, 13].

The classical stochastic averaging method can only be applied to s.d.o.f. strongly non-linear systems. In recent years, the stochastic averaging method for quasi-Hamiltonian systems has been developed [14–16]. The method can be applied to m.d.o.f. integrable, partially integrable and completely integrable Hamiltonian systems subject to lightly linear and (or) non-linear dampings and external and (or) parametric excitations of Gaussian white noises. It has been shown that the number of averaged Itô equations (or the dimension of the averaged FPK equation) is equal to the number of independent integrals of motion in involution and the number of resonant relations of the associated Hamiltonian systems. The technique for obtaining the exact stationary solution to the averaged FPK equations was also developed. The stochastic averaging method for quasi-Hamiltonian systems has been applied to study the stochastic stability, stochastic bifurcation, reliability and optimal non-linear stochastic control of quasi-Hamiltonian systems [16–26].

Engineering systems are often subjected to combined harmonic and random excitations. The classical stochastic averaging method has been applied to linear conservative oscillators subject to light damping and combined harmonic and wideband random excitations to obtain the conditions of moment stability [27–31] or to obtain the stationary probability density [32, 33]. To the authors' knowledge, there is no stochastic averaging method available which can be applied to non-linear conservative oscillators subject to light damping and combined harmonic and random excitations.

In the present paper, the stochastic averaging method for s.d.o.f. strongly non-linear oscillators with light damping under external and (or) parametric excitations of both harmonic and white-noise forces is developed. The method is then applied to a Duffing oscillator with hardening stiffness under both external harmonic excitation and external and parametric excitations of white noises. The averaged FPK equation is solved by using the path integration technique. The stationary joint probability density of amplitude and phase obtained by using the stochastic averaging and path integration is used to examine the stochastic jump of the Duffing oscillator under combined harmonic and white-noise excitations and its bifurcation as the system parameters change.

2. GENERALIZED HARMONIC FUNCTIONS

Consider the free vibration of a non-linear conservative oscillator whose equation of motion is of the form

$$\ddot{x} + g(x) = 0. \tag{1}$$

The first integral (energy integral) of oscillator (1) is

$$\tfrac{1}{2}\dot{x}^2 + v(x) = H, \tag{2}$$

where H is the total energy of the oscillator while

$$v(x) = \int_0^x g(u)\,\mathrm{d}u \tag{3}$$

is the potential energy. $g(x)$ and $v(x)$ are assumed to satisfy the following conditions: (i) $g(0) = 0$; (ii) all the trajectories in domain U of phase plan $(x, \dot{x})$ are periodic surrounding (0, 0). The periodic solution of equation (1) in U can be written as [34]

$$x(t) = a\cos\varphi(t) + b, \tag{4}$$

$$\dot{x}(t) = -\,a\nu(a, \varphi)\sin\varphi(t), \tag{5}$$

where

$$\varphi(t) = \tau(t) + \theta, \tag{6}$$

$$\nu(a, \varphi) = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{\frac{2[v(a+b) - v(a\cos\varphi + b)]}{a^2\sin^2\varphi}}, \tag{7}$$

a and b are constants and related to H as follows

$$v(a+b) = v(-\,a+b) = H, \tag{8}$$

$\cos\varphi(t)$ and $\sin\varphi(t)$ are called generalized harmonic functions [34]. Obviously, a is the amplitude of oscillation, $\nu(a, \varphi)$ is the instantaneous frequency of oscillation and θ is the phase angle. Expanding ν^{-1} into Fourier series

$$\nu^{-1}(a, \varphi) = C_0(a) + \sum_{n=1}^{\infty} C_n(a)\cos n\varphi \tag{9}$$

and then integrating equation (7) with respect to τ yield

$$t = C_0(a)\tau + \sum_{n=1}^{\infty} \frac{1}{n} C_n(a)\sin n\varphi. \tag{10}$$

Further integrating equation (10) with respect to τ from 0 to 2π leads to the averaged period

$$T(a) = 2\pi C_0(a) \tag{11}$$

and the averaged frequency

$$\omega(a) = \frac{1}{C_0(a)} \tag{12}$$

of the oscillator. The integration constants in equations (10)–(12) have been neglected. Note that Fourier series in equations (9) and (10) converge rapidly and can be approximated with a few terms in practical calculation.

3. STOCHASTIC AVERAGING PROCEDURE

Consider the response of oscillator (1) subject to light damping and external and (or) parametric excitations of both harmonic and white-noise forces. The equation of motion of the system is of the form

$$\ddot{X} + g(X) = \varepsilon f(X, \dot{X}, \Omega t) + \varepsilon^{1/2} h_k(X, \dot{X}) W_k(t), \quad k = 1, 2, \ldots, m, \tag{13}$$

where ε is a small parameter, $f(X, \dot{X}, \Omega t)$ represents linear and (or) non-linear dampings and external and (or) parametric harmonic excitation with frequency Ω, $h_k(X, \dot{X}) W_k(t)$ represent external and (or) parametric excitations of Gaussian white noises $W_k(t)$ with intensities $2D_{kl}$.

The system governed by equation (13) without white-noise excitation has been studied by Xu and Cheung [34]. In the case of resonance, the response is a harmonic motion. If the added white-noise excitation does not destabilize the system, then its response will be random spread of periodic motion, i.e., periodic non-stationary process [4]. Thus, the solution of system (13) in U is assumed to be of the following form:

$$X(t) = A\cos\Phi(t) + B, \tag{14}$$

$$\dot{X}(t) = -Av(A, \Phi)\sin\Phi(t), \tag{15}$$

where

$$\Phi(t) = \tau(t) + \Theta(t), \tag{16}$$

$$v(A, \Phi) = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{\frac{2[v(A+B) - v(A\cos\Phi + B)]}{A^2\sin^2\Phi}} \tag{17}$$

and A, B, Φ, Θ, τ and v are all random processes. Differentiating equation (14) with respect to t and equating the resultant to equation (15) yield

$$\dot{A}(\cos\Phi + h) - \dot{\Theta} A\sin\Phi = 0, \tag{18}$$

where

$$h = \frac{\mathrm{d}B}{\mathrm{d}A} = \frac{g(-A+B) + g(A+B)}{g(-A+B) - g(A+B)}. \tag{19}$$

Differentiating equation (15) with respect to t and substituting the resultant into equation (13) leads to

$$\dot{A}\left\{v(A, \Phi)\sin\Phi] + A\frac{\partial}{\partial A}[v(A, \Phi)\sin\Phi]\right\} + \dot{\Theta}\frac{\partial}{\partial\Phi}[v(A, \Phi)\sin\Phi]$$
$$= -\varepsilon f(A\cos\Phi + b, -Av(A, \Phi)\sin\Phi, \Omega t) - \varepsilon^{1/2} h_k(A\cos\Phi + B, -Av(A, \Phi)\sin\Phi) W_k(t). \tag{20}$$

Solving equations (18) and (20), one obtains

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \varepsilon F_1(A, \Phi, \Omega t) + \varepsilon^{1/2} H_{1k}(A, \Phi) W_k(t),$$
$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \varepsilon F_2(A, \Phi, \Omega t) + \varepsilon^{1/2} H_{2k}(A, \Phi) W_k(t), \tag{21}$$

where

$$F_1 = \frac{A}{g(A+B)(1+h)} f(A\cos\Phi + B, \ -Av(A, \Phi)\sin\Phi, \Omega t) v(A, \Phi)\sin\Phi,$$
$$F_2 = -\frac{A}{g(A+B)(1+h)} f(A\cos\Phi + B, \ -Av(A, \Phi)\sin\Phi, \Omega t) v(A, \Phi)(\cos\Phi + h),$$
$$H_{1k} = \frac{A}{g(A+B)(1+h)} h_k(A\cos\Phi + B, \ -Av(A, \Phi)\sin\Phi, \Omega t) v(A, \Phi)\sin\Phi, \tag{22}$$
$$H_{2k} = \frac{1}{g(A+B)(1+h)} h_k(A\cos\Phi + B, \ -Av(A, \Phi)\sin\Phi, \Omega t) v(A, \Phi)(\cos\Phi + h).$$

Equation (21) can be modelled as Stratonovich stochastic differential equation and then transformed into Itô stochastic differential equation by adding Wong–Zakai correction terms [35]. The result is

$$\mathrm{d}A = \varepsilon m_1(A, \Phi, \Omega t)\,\mathrm{d}t + \varepsilon^{1/2}\sigma_{1r}(A, \Phi)\,\mathrm{d}B_r(t),$$
$$\mathrm{d}\Theta = \varepsilon m_2(A, \Phi, \Omega t)\,\mathrm{d}t + \varepsilon^{1/2}\sigma_{2r}(A, \Phi)\,\mathrm{d}B_r(t), \tag{23}$$
$$r = 1, 2, \ldots, 2m,$$

where $B_r(t)$ are unit Wiener processes,

$$m_i = F_i + D_{kl}\frac{\partial H_{ik}}{\partial A} H_{1l} + D_{kl}\frac{\partial H_{ik}}{\partial \Phi} H_{2l},$$
$$b_{ij} = \sum_{r,s=1}^{2m} \sigma_{ir}\sigma_{js} = 2D_{kl}H_{ik}H_{jl}, \tag{24}$$
$$i, j = 1, 2, \quad k, l = 1, 2, \ldots, m, \quad r, s = 1, 2, \ldots, 2m.$$

System (13) has harmonic excitation and so two cases can be identified: resonant case and non-resonant case. The resonant case is more interesting and considered in the following. Assume that

$$\frac{\Omega}{\omega(a)} = \frac{q}{p} + \varepsilon\sigma, \tag{25}$$

where p and q are relatively prime positive integers and $\sigma = O(1)$ is the detuning parameter. In this case equation (10) becomes

$$\Omega t = \frac{q}{p}\Phi + \varepsilon\sigma\tau - \frac{q}{p}\Theta + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(A)\sin n\Phi. \tag{26}$$

Introduce new variable Γ such that

$$\Gamma = \varepsilon\sigma\tau - \frac{q}{p}\Theta. \tag{27}$$

Then equation (26) can be rewritten as

$$\Omega t = \Psi + \Gamma, \tag{28}$$

where

$$\Psi = \Psi(A, \Phi) = \frac{q}{p}\Phi + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(A)\sin n\Phi \tag{29}$$

with the transformation from Θ to Γ as defined by equation (27), equation (23) can be rewritten as

$$\begin{aligned} &\mathrm{d}A = \varepsilon m_1(A, \Phi, \Psi + \Gamma)\,\mathrm{d}t + \varepsilon^{1/2}\sigma_{1r}(A, \Phi)\,\mathrm{d}B_r(t), \\ &\mathrm{d}\Gamma = \left[\varepsilon m_2(A, \Phi, \Psi + \Gamma)\left(-\frac{q}{p}\right) + \left(\frac{\Omega}{\omega(a)} - \frac{q}{p}\right)v(A, \Phi)\right]\mathrm{d}t - \varepsilon^{1/2}\frac{q}{p}\sigma_{2r}(A, \Phi)\,\mathrm{d}B_r(t), \\ &\quad r = 1, 2, \ldots, 2m. \end{aligned} \tag{30}$$

The drift and diffusion coefficients in Itô equation (30) are functions of slowly varying processes A and Γ and rapidly varying process Φ. Averaging them with respect to Φ yields the following averaged Itô equations:

$$\begin{aligned} &\mathrm{d}A = \varepsilon\bar{m}_1(A, \Gamma)\,\mathrm{d}t + \varepsilon^{1/2}\bar{\sigma}_{1r}(A)\,\mathrm{d}B_r(t), \\ &\mathrm{d}\Gamma = \varepsilon\bar{m}_2(A, \Gamma)\,\mathrm{d}t + \varepsilon^{1/2}\bar{\sigma}_{2r}(A)\,\mathrm{d}B_r(t), \quad r = 1, 2, \ldots, 2m, \end{aligned} \tag{31}$$

where

$$\begin{aligned} &\bar{m}_1 = \langle m_1(A, \Phi, \Psi + \Gamma)\rangle_\Phi, \\ &\bar{m}_2 = \left\langle m_2(A, \Phi, \Psi + \Gamma)\left(-\frac{q}{p}\right) + \frac{v(A, \Phi)}{\varepsilon}\left(\frac{\Omega}{\omega(a)} - \frac{q}{p}\right)\right\rangle_\Phi, \\ &\bar{b}_{ij} = \sum_{r,s=1}^{2m}\bar{\sigma}_{ir}\bar{\sigma}_{js} = \langle b_{ij}\rangle_\Phi = \left\langle\sum_{r,s=1}^{2m}\sigma_{ir}\sigma_{js}\right\rangle_\Phi, \quad i, j = 1, 2, \quad r, s = 1, 2, \ldots, 2m \end{aligned} \tag{32}$$

and $\langle\ \rangle_\Phi$ represents the averaging with respect to Φ from 0 to 2π.

The averaged FPK equation associated with averaged Itô equation (31) is

$$\frac{\partial p}{\partial t}=\varepsilon\left[-\frac{\partial}{\partial a}(\bar{m}_1 p)-\frac{\partial}{\partial \gamma}(\bar{m}_2 p)+\frac{1}{2}\frac{\partial^2}{\partial a^2}(\bar{b}_{11}p)+\frac{\partial^2}{\partial a\,\partial\gamma}(\bar{b}_{12}p)+\frac{1}{2}\frac{\partial^2}{\partial\gamma^2}(\bar{b}_{22}p)\right], \tag{33}$$

where $p=p(a,\gamma,t\,|\,a_0,\gamma_0)$ is the transition probability density of amplitude A and phase Γ. The initial condition of FPK equation (33) is

$$p=\delta(a-a_0)\delta(\gamma-\gamma_0),\quad t=0. \tag{34}$$

Since $p(a,\gamma,t\,|\,a_0,\gamma_0)$ is a periodic function of γ, the boundary condition of FPK equation (33) with respect to γ is

$$p(a,\gamma+2n\pi,t\,|\,a_0,\gamma_0)=p(a,\gamma,t\,|\,a_0,\gamma_0). \tag{35}$$

As for the boundary condition of FPK equation (33) with respect to a, one is

$$p=\text{finite at } a=0. \tag{36}$$

That means $a=0$ is a reflecting boundary. The other boundary condition depends on the behavior of the non-linear oscillator in equation (13) with $\varepsilon=0$. In the simplest case, where all the solutions of oscillator (13) with $\varepsilon=0$ in whole phase plane $(x,\dot{x})$ are periodic surrounding (0, 0), the other boundary condition of FPK equation (33) with respect to a is

$$p,\ \frac{\partial p}{\partial a}\to 0,\quad \text{as } a\to\infty. \tag{37}$$

4. AVERAGED EQUATION FOR DUFFING OSCILLATOR

As an application of the stochastic averaging procedure developed in the last section, in this section we derive the averaged equations for a Duffing oscillator with hardening stiffness subject to lightly linear damping and combined externally harmonic excitation and externally and parametrically white-noise excitations. The equation of motion of the system is of the form

$$\ddot{X}+\omega^2X+\alpha X^3=-\beta\dot{X}+E\cos\Omega t+W_1(t)+XW_2(t), \tag{38}$$

where α, β and E are positive constants representing the strength of non-linearity, the coefficient of damping and the amplitude of harmonic excitation, respectively, $W_1(t)$ and $W_2(t)$ are independent Gaussian white noises in the sense of Stratonovich with intensity $2D_1$ and $2D_2$ respectively. β, E and D_i are assumed to be of the same order of ε. For such a Duffing oscillator,

$$g(x)=\omega^2x+\alpha x^3,\qquad v(x)=\omega^2x^2/2+\alpha x^4/4,\qquad b=h=0 \tag{39}$$

and

$$v^{-1}(a,\varphi)=[(\omega^2+3\alpha a^2/4)(1+\lambda\cos 2\varphi)]$$

$$=\sum_{n=0}^{\infty}C_{2n}(a)\cos 2n\varphi,$$

$$C_{2n}(a) = \frac{1}{2\pi}\int_0^{2\pi} v^{-1}(a, \varphi)\cos 2n\varphi \,\mathrm{d}\varphi, \tag{40}$$

$$\omega(a) = 1/C_0(a),$$

$$\lambda = \tfrac{1}{4}\alpha A^2/(\omega^2 + \tfrac{3}{4}\alpha A^2).$$

Consider the case of primary resonance

$$\Omega/\omega(a) = 1 + \sigma, \tag{41}$$

where σ is of the same order of ε. Introducing transformations

$$X(t) = A\cos\Phi(t), \qquad \dot{X}(t) = -Av(A, \Phi)\sin\Phi, \tag{42}$$

where Φ is defined by equation (16), and following the derivation from equation (18) to equation (21), one obtains

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= f_1(A, \Phi, \Omega t) + g_{11}(A, \Phi)W_1(t) + g_{12}(A, \Phi)W_2(t),\\ \frac{\mathrm{d}\Theta}{\mathrm{d}t} &= f_2(A, \Phi, \Omega t) + g_{21}(A, \Phi)W_1(t) + g_{22}(A, \Phi)W_2(t),\end{aligned} \tag{43}$$

where

$$\begin{aligned}f_1 &= -\frac{A}{g(A)}[\beta A v(A, \Phi)\sin\Phi + E\cos\Omega t]\,v(A, \Phi)\sin\Phi,\\ f_2 &= -\frac{1}{g(A)}[\beta A v(A, \Phi)\sin\Phi + E\cos\Omega t]\,v(A, \Phi)\cos\Phi,\\ g_{11} &= -\frac{A}{g(A)}v(A, \Phi)\sin\Phi, \qquad g_{12} = -\frac{A^2}{g(A)}v(A, \Phi)\sin\Phi\cos\Phi,\\ g_{21} &= -\frac{1}{g(A)}v(a, \Phi)\cos\Phi, \qquad g_{22} = -\frac{A}{g(A)}v(A, \Phi)\cos^2\Phi.\end{aligned} \tag{44}$$

Equation (43) can be modelled as the following Itô stochastic differential equation by adding Wong–Zakai correction terms [35]

$$\begin{aligned}\mathrm{d}A &= a_1(A, \Phi, \Omega t)\,\mathrm{d}t + \eta_{1r}(A, \Phi)\,\mathrm{d}B_r(t),\\ \mathrm{d}\Theta &= a_2(A, \Phi, \Omega t)\,\mathrm{d}t + \eta_{2r}(A, \Phi)\,\mathrm{d}B_r(t), \qquad r = 1, 2,\end{aligned} \tag{45}$$

where

$$\begin{aligned}a_i &= f_i + D_k\left(g_{1k}\frac{\partial g_{ik}}{\partial A} + g_{2k}\frac{\partial g_{ik}}{\partial \Phi}\right),\\ b_{ij} &= \sum_{r,s=1}^{2}\eta_{ir}\eta_{js} = 2D_k g_{ik}g_{jk}, \qquad i, j, k = 1, 2.\end{aligned} \tag{46}$$

As in equation (27), introducing new variable

$$\Gamma = \sigma\tau - \Theta \tag{47}$$

equation (45) is transformed into

$$\begin{aligned} \mathrm{d}A &= a_1(A, \Phi, \Psi + \Gamma)\,\mathrm{d}t + \eta_{1r}(A, \Phi)\,\mathrm{d}B_r(t), \\ \mathrm{d}\Gamma &= [-a_2(A, \Phi, \Psi + \Gamma) + (\Omega/\omega(a) - 1)v(A, \Phi)]\,\mathrm{d}t \\ &\quad - \eta_{2r}(A, \Phi)\,\mathrm{d}B_r(t), \qquad r = 1, 2. \end{aligned} \tag{48}$$

Finally, applying the deterministic averaging to equation (48) with respect to Φ, one obtains

$$\begin{aligned} \mathrm{d}\bar{A} &= \bar{a}_1(A, \Gamma)\,\mathrm{d}t + \bar{\eta}_{1r}(A)\,\mathrm{d}B_r(t), \\ \mathrm{d}\Gamma &= \bar{a}_2(A, \Gamma)\,\mathrm{d}t + \bar{\eta}_{2r}(A)\,\mathrm{d}B_r(t), \qquad r = 1, 2, \end{aligned} \tag{49}$$

where

$$\begin{aligned} \bar{a}_1 = &-\beta A(\omega^2 + 5\alpha A^2/8)/2(\omega^2 + \alpha A^2) + E \sin\Gamma \Big\langle v(A, \Phi)\sin\Phi \\ &\times \sin\Big(\Phi + \Omega \sum_{n=1}^{\infty} \frac{1}{n} C_n(a) \sin n\Phi\Big)\Big\rangle_{\Phi} \Big/ (\omega^2 + \alpha A^2) \\ &- \alpha D_1 A(3\omega^2 + 3\alpha A^2/2)/4(\omega^2 + \alpha A^2)^3 + D_1(\omega^2 + 7\alpha A^2/8)/2A(\omega^2 + \alpha A^2)^2 \\ &+ D_2\omega^2 A(\omega^2 + \alpha A^2/2)/8(\omega^2 + \alpha A^2)^3 + D_2 A(\omega^2 + 7\alpha A^2/8)/4(\omega^2 + \alpha A^2)^2, \\ \bar{a}_2 = &\; E\cos\Gamma \Big\langle v(A, \Phi)\cos\Phi \cos\Big(\Phi + \Omega \sum_{n=1}^{\infty} \frac{1}{n} C_n(A)\sin n\Phi\Big)\Big\rangle_{\Phi} \Big/ A(\omega^2 + \alpha A^2) \\ &+ [\Omega C_0(A) - 1]\langle v(A, \Phi)\rangle_{\Phi}, \\ \bar{b}_{11} = &\sum_{r,s=1}^{2} \bar{\eta}_{1r}\bar{\eta}_{1s} \\ = &\; D_1(\omega^2 + 5\alpha A^2/8)/(\omega^2 + \alpha A^2)^2 + D_2 A^2(\omega^2 + 3\alpha A^2/4)/4(\omega^2 + \alpha A^2)^2, \\ \bar{b}_{22} = &\sum_{r,s=1}^{2} \bar{\eta}_{2r}\bar{\eta}_{2s} \\ = &\; D_1(\omega^2 + 7\alpha A^2/8)/A^2(\omega^2 + \alpha A^2)^2 + 2D_2(3\omega^2/8 + 11\alpha A^2/32)/(\omega^2 + \alpha A^2)^2, \\ \bar{b}_{12} = &\; \bar{b}_{21} = \sum_{r,s=1}^{2} \sigma_{1r}\sigma_{2s} = 0. \end{aligned} \tag{50}$$

In the derivation of $\bar{a}_1$, the following formula has been used:

$$\frac{\mathrm{d}}{\mathrm{d}A}\left(\frac{A}{g(A)}\right) = -\frac{2\alpha A}{(\omega^2 + \alpha A^2)^2}. \tag{51}$$

The averaged FPK equation associated with Itô equation (49) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial a}(\bar{a}_1 p) - \frac{\partial}{\partial \gamma}(\bar{a}_2 p) + \frac{1}{2}\frac{\partial^2}{\partial a^2}(\bar{b}_{11} p) + \frac{1}{2}\frac{\partial^2}{\partial \gamma^2}(\bar{b}_{22} p). \tag{52}$$

The initial condition is the same as in equation (34) and the boundary conditions are the same as in equations (35)–(37).

In the special case of $\alpha = D_2 = 0$, system (38) is linear and equation (49) is reduced to

$$\begin{aligned} \mathrm{d}A &= (-\beta A/2 + E\sin\Gamma/2\omega + D_1/2\omega^2 a)\,\mathrm{d}t + (D_1^{1/2}/\omega)\,\mathrm{d}B_1(t), \\ \mathrm{d}\Gamma &= (E\cos\Gamma/2\omega A + (\Omega - \omega))\,\mathrm{d}t + (D_1^{1/2}/\omega A)\,\mathrm{d}B_2(t). \end{aligned} \tag{53}$$

Applying the solution procedure proposed in reference [33], one obtains the following exact stationary solution to the averaged FPK equation:

$$p(a, \gamma) = C\exp[-(\beta\omega^2/2D_1)a^2 + (Ea/\omega)(s\cos\gamma + D_1\sin\gamma/\omega^2)/(s^2 + (D_1/\omega^2)^2], \tag{54}$$

where C is a normalization constant and

$$s = 2D_1(\omega - \Omega)/\beta\omega^2. \tag{55}$$

5. SOLUTION OF AVERAGED FPK EQUATION

Averaged FPK equation (33) is a two-dimensional linear elliptic partial differential equation with variable coefficients and generally can be solved only numerically. The method of path integration is one such numerical procedure and it is appropriate for the present purpose. Early application of the path integration to solving FPK equation was made by Wehner and Wolfer [36]. Recent improvements of the technique can be found in references [37, 38].

$[A, \Gamma]^{\mathrm{T}}$ in equation (31) is a two-dimensional vector diffusive Markov process. Dividing time interval $[0, t]$ into N short subintervals of length τ, the solution to FPK equation (33) with initial condition (34) can be represented as

$$p(a, \gamma, t) = \prod_{i=1}^{N} \int_V p(a^{(i)}, \gamma^{(i)}, \tau | a^{(i-1)}, \gamma^{(i-1)})\,\mathrm{d}a^{(i-1)}\,\mathrm{d}\gamma^{(i-1)}\,p(a_0, \gamma), \tag{56}$$

where $t_N = t$, $a^{(N)} = a$, $\gamma^{(N)} = \gamma$; $p(a^{(i)}, \gamma^{(i)}, \tau | a^{(i-1)}, \gamma^{(i-1)})$ is the so-called short time transition probability density (STTPD); V is the domain of plane (a, γ) defined by the problem in hand. In the method of path integration, STTPD is approximated by a Gaussian probability density. Several different forms of Gaussian probability density have been proposed for STTPD and a simple one in reference [36] is used in the present paper. For a fixed point $(a^{(i)}, \gamma^{(i)})$ at time t_i, the point $(a^{(i-1)}, \gamma^{(i-1)})$ at time t_{i-1} is determined by using the Runge–Kutta–Maruyma approximation [38].

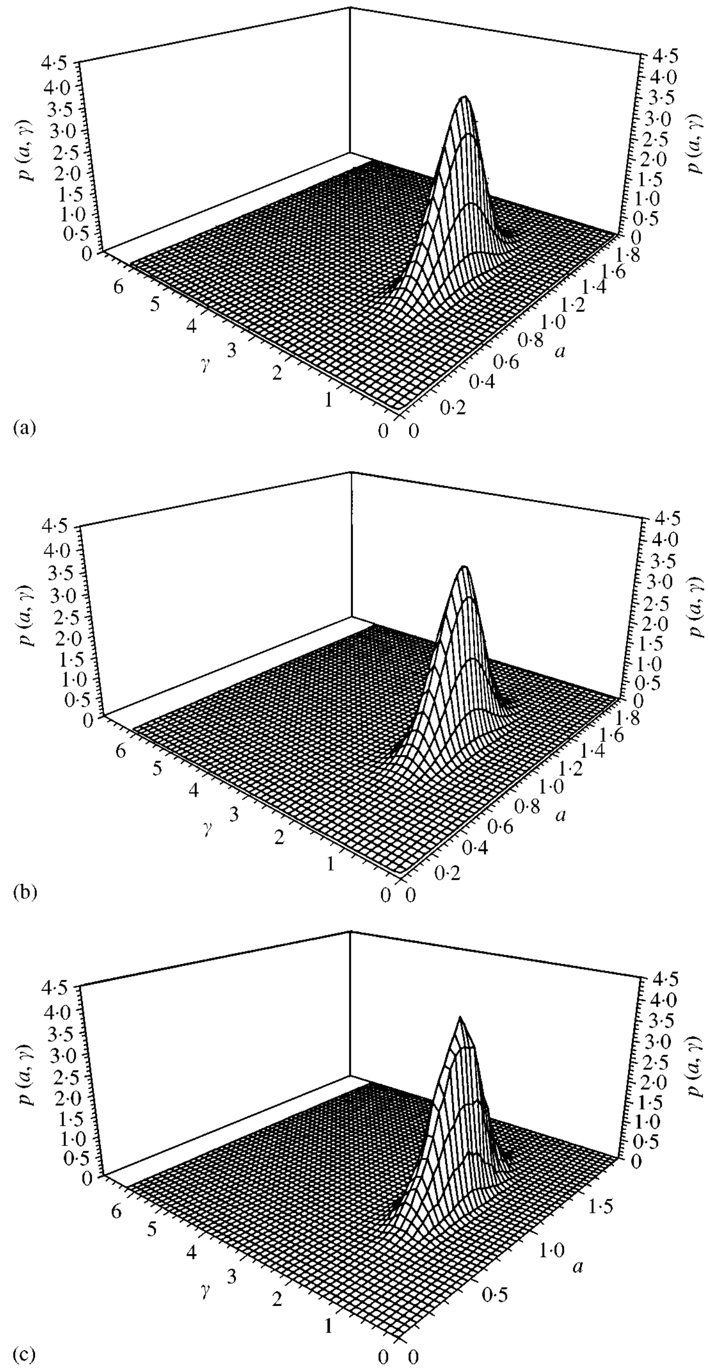


Figure 1. Stationary probability density $p(a, \gamma)$ of degenerated linear oscillator under combined harmonic and white-noise excitations. $\alpha = 0$, $\beta = 0{\cdot}1$, $\omega = \Omega = 1{\cdot}0$, $E = 0{\cdot}1$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) exact stationary solution (58) of averaged FPK equation; (c) from the digital simulation of original equation of motion.

The computer program of the path integration developed by the present authors is first used to calculate the stationary solution of the averaged FPK equation associated with Itô equation (53). The result is shown in Figure 1(a). To check the accuracy of the stochastic averaging and path integration, the exact stationary solution (54) of the averaged FPK equation and the digital simulation of original equation (38) with $\alpha = 0$ are also obtained and are shown in Figure 1(b) and 1(c) respectively. It is seen from Figure 1 that the stochastic averaging and path integration together yield satisfactory result.

6. STOCHASTIC JUMP AND BIFURCATION OF DUFFING OSCILLATOR

It is well known that a Duffing oscillator with hardening stiffness subject to harmonic excitation may exhibit the phenomenon of sharp jumps in amplitude [39]. The jump phenomenon may also occur when the Duffing oscillator is subjected to narrowband random excitation [40] and this stochastic jump phenomenon was the subject of many publications since early 1960s (see, for example, the list of references in reference [41]). To the author's knowledge, the jump phenomenon of Duffing oscillator under combined harmonic and white-noise excitations has not been reported in the literature. In this section, this stochastic jump and its bifurcation as the system parameters change are examined

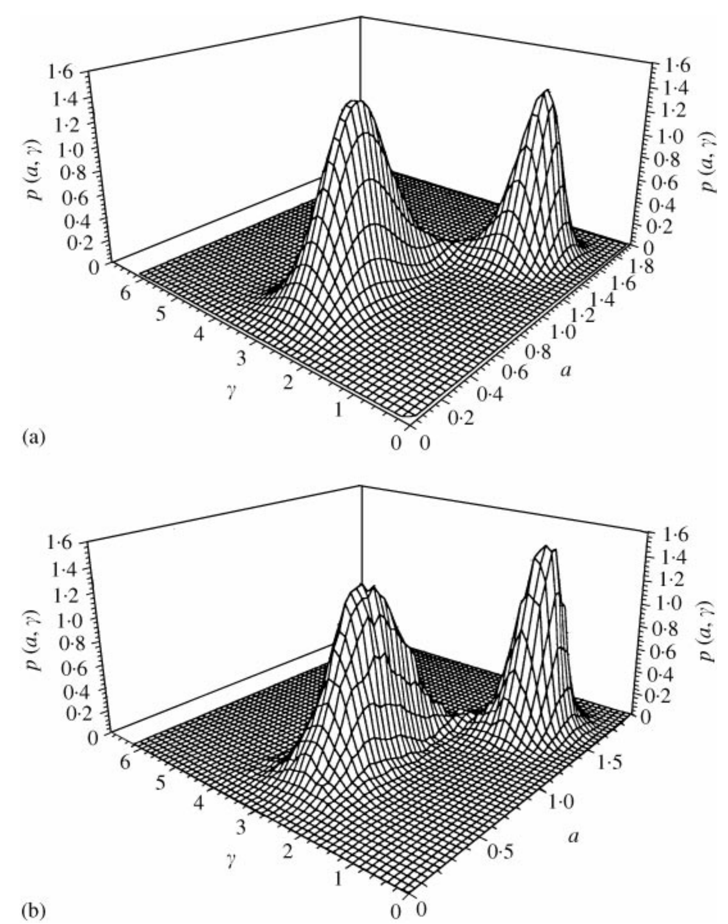
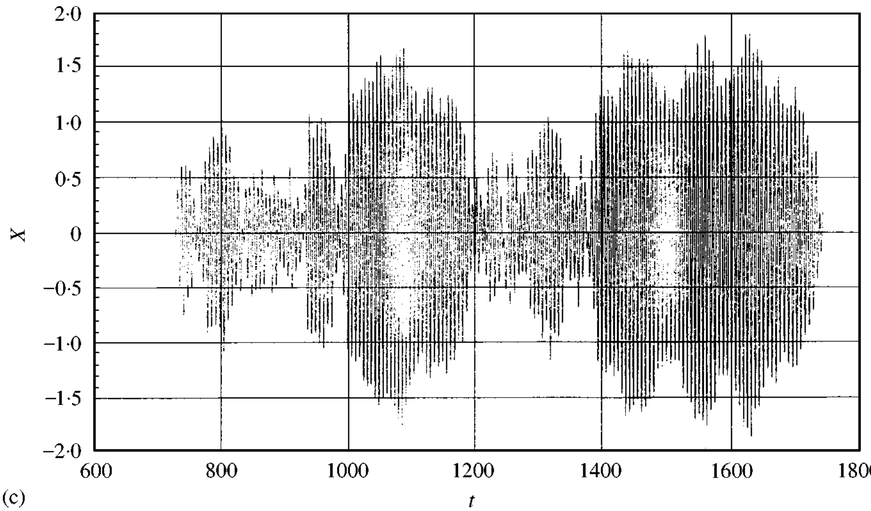
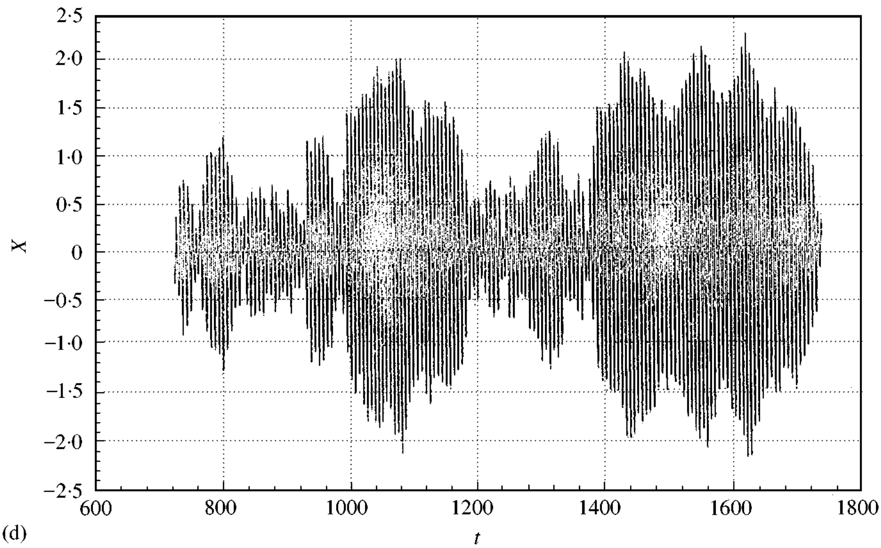


Figure 2. Stationary probability density $p(a, \gamma)$ ((a) and (b)) and sample functions ((c) and (d)) of system (38), $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38); (c) displacement; (d) velocity.



(c)



(d)

Figure 2. Continued.

based on the stationary solution to averaged FPK equation (52) obtained by using the path integration technique and the results from the digital simulation of original equation (38).

The deterministic jump behavior of a Duffing oscillator under pure harmonic excitation is associated with the fact that, over a range of the values of the ratio of excitation frequency to the natural frequency of the degenerated linear oscillator, the amplitude response is triple-valued. Among the three values of amplitude, two are stable while the other is unstable. Jump occurs between the two stable amplitudes as the frequency ratio changes slowly and passes through the extreme values of the frequency ratio interval of triple-valued amplitude. So, amplitude response curve (amplitude versus frequency ratio) is enough to study the deterministic jump phenomenon. The stochastic jump of Duffing oscillator under narrowband random excitation, on the other hand, is essentially a transition of the response from one more probable motion to another or *vice versa* [41]. A more probable motion is represented by a peak of the stationary joint probability density of displacement and velocity (or, of amplitude and phase). Jumps may occur when the probability density has at least two peaks (bimodal). So, the stochastic jump phenomena can be examined by using the stationary joint probability density of displacement and velocity (or, of amplitude and phase). It should be noted that the variance of a stationary response is unique and the stochastic jump phenomena cannot be explained by using triple-valued variance. In the following, it will be shown that the stochastic jump of a Duffing oscillator under combined harmonic and white-noise excitations can be examined in a similar way.

A typical response exhibiting stochastic jump phenomenon of a Duffing oscillator under combined harmonic and white-noise excitations is shown in Figure 2, where (a) and (b) are the stationary joint probability densities of amplitude and phase obtained from solving the averaged FPK equation (52) by using the path integration technique and from the digital simulation of original equation (38), respectively, and (c) and (d) are the sample functions of the displacement and velocity, respectively, from the digital simulation. It is is seen that the two probability densities are in good agreement and they are both bimodal. It implies that there are two more probable motions in the response of the Duffing oscillator and stochastic jumps may occur. This is verified by the sample functions of displacement and velocity. Taking expectation of equation (38) leads to the equation for the same Duffing oscillator under pure harmonic excitation. So, it can be expected that the two peaks of the stationary probability density are located on the amplitude response curve of the Duffing oscillator under pure harmonic excitation, as indicated by curve B in Figure 3. On the other hand, the stochastic jumps of a Duffing oscillator under combined harmonic and white-noise excitations can be regarded as random spread of the deterministic jump of the same oscillator under pure harmonic excitation. However, the deterministic and stochastic jump phenomena have significant difference. The deterministic jump occurs only at the two extreme values of the frequency ratio interval to triple-valued amplitude. The jump from upper branch to lower branch of amplitude response curve occurs only at the right extreme

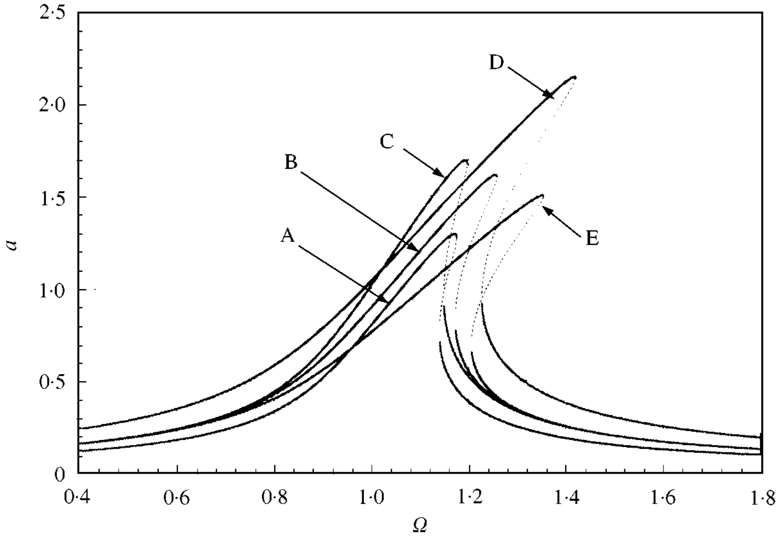


Figure 3. Amplitude response curve of Duffing oscillator under pure harmonic excitation, curve A: $\omega = 1{\cdot}0$, $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $E = 0{\cdot}15$; curve B: $\omega = 1{\cdot}0$, $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $E = 0{\cdot}2$; curve C: $\omega = 1{\cdot}0$, $\alpha = 0{\cdot}2$, $\beta = 0{\cdot}1$, $E = 0{\cdot}2$; curve D: $\omega = 1{\cdot}0$, $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $E = 0{\cdot}3$; curve E: $\omega = 1{\cdot}0$, $\alpha = 0{\cdot}5$, $\beta = 0{\cdot}1$, $E = 0{\cdot}2$.

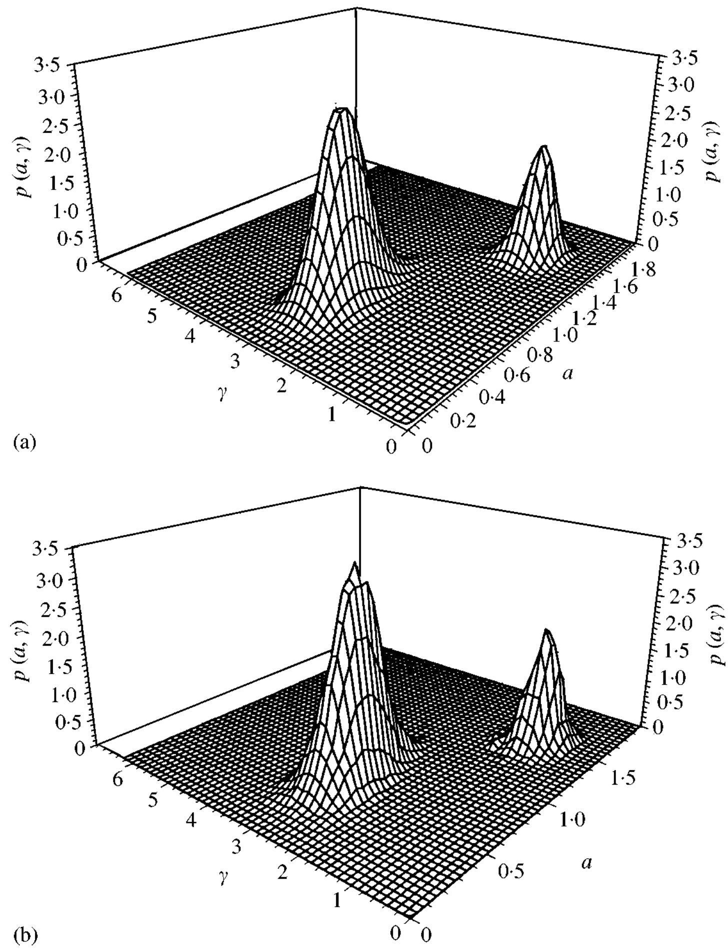


Figure 4. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}002$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

value while the backward jump occurs only at the left extreme value. On the other hand, the stochastic jump may occur at any frequency ratio within this interval and jumps may occur back and forth.

As in the deterministic case, whether stochastic jump occurs depends on the system's parameters, such as the intensity of white noise, the frequency ratio, the amplitude of harmonic excitation and the strength of non-linearity. Since the occurence of stochastic jump is related to bimodal probability density, we call the appearance or disappearance of stochastic jump as the system's parameters change the bifurcation of the stochastic jump. In the following, we will examine the bifurcation of the stochastic jump as the system's parameters change.

First, let us examine the effect of the intensity of white noise on the stochastic jump. In Figures 4 and 5 are shown the joint probability densities of the same system as that shown in Figure 2 except the intensity of white noise. In all three cases shown in Figure 2, 4 and 5, the joint probability densities are bimodal and so jumps may occur, however, there is still

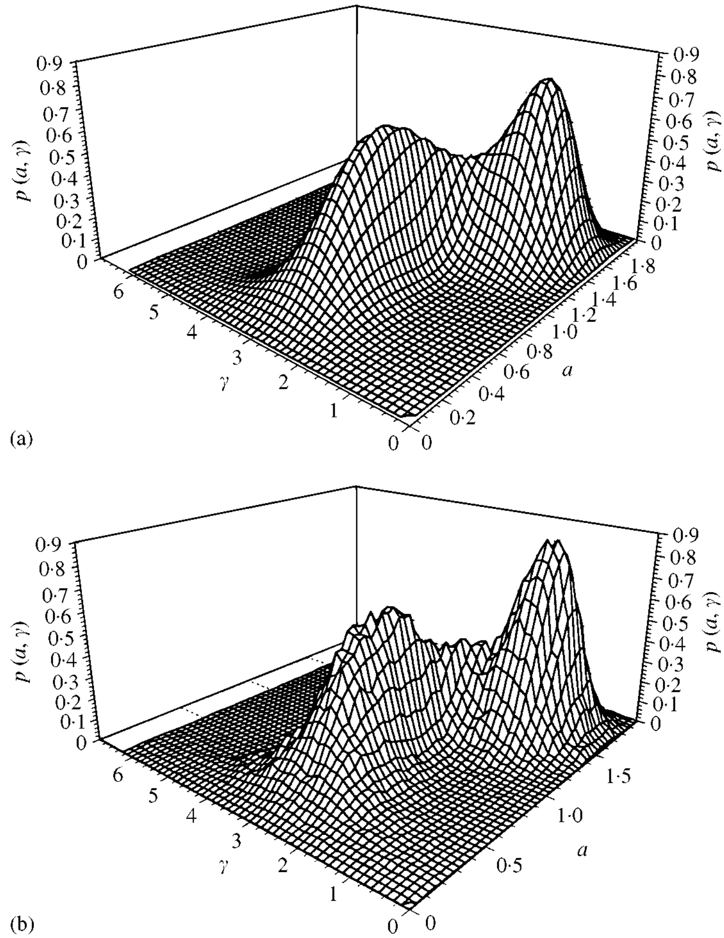


Figure 5. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}01$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

slight difference among them. At smaller intensity, the two peaks are more well separated, jumps occur more rarely and the most probable motion is around the lower branch of the amplitude response curve. At higher intensity, on the other hand, the two peaks connect and even merge, jumps occur more frequently and the most probable motion is around the upper branch of the amplitude response curve. It can be expected that as the intensity of white noise approaches to zero, the stochastic jump approaches the deterministic jump.

Second, let us examine the effect of the frequency ratio on the stochastic jump. In Figures 6 and 7 are shown the joint probability densities of the same system as that shown in Figure 2 except the frequency ratio. They are both unimodal and so no jump may occur. This is because the values of frequency ratio in these two cases are well off the interval of triple-valued amplitude, see curve B in Figure 3. The two joint probability densities in Figures 6 and 7 represent the most probable motion around the lower and upper branches of the amplitude response curve respectively. Thus, the bifurcation of the stochastic jump of

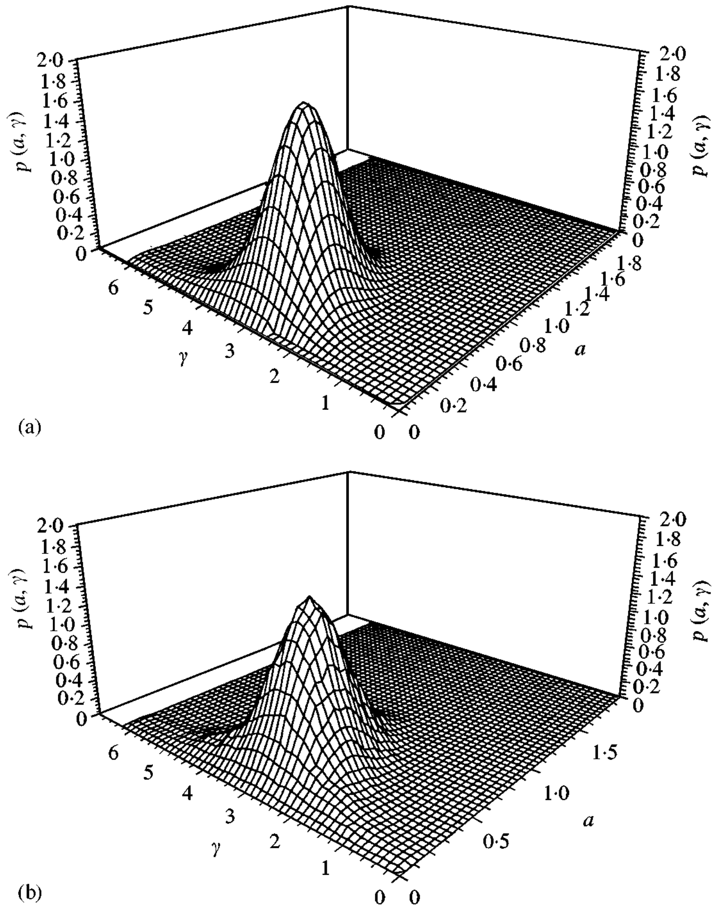


Figure 6. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}3$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

a Duffing oscillator under combined harmonic and white-noise excitations as the frequency ratio changes can be roughly estimated based on the amplitude response curve of the same oscillator under pure-harmonic excitation.

Third, let us consider the effect of the amplitude of harmonic excitation on the stochastic jump. Two joint probability densities for the same system as that shown in Figure 2 except the amplitude of harmonic excitation are shown in Figures 8 and 9. They are both unimodal and so no jump may occur in these cases. This is also because these two cases shown in Figures 8 and 9 are well off the frequency ratio interval of triple-valued amplitude, see curves A and D in Figure 3. So, the bifurcation of the stochastic jump of a Duffing oscillator under combined harmonic and white-noise excitations as the amplitude of harmonic excitation changes can also be roughly estimated based on the amplitude response curves of the same oscillator under pure harmonic excitation with different excitation amplitudes.

Finally, consider the effect of the strength of non-linearity on the stochastic jump. The two joint probability densities for the same system as that shown in Figure 2 except the strength of non-linearity are shown in Figure 10 and 11. They are unimodal and so no jump

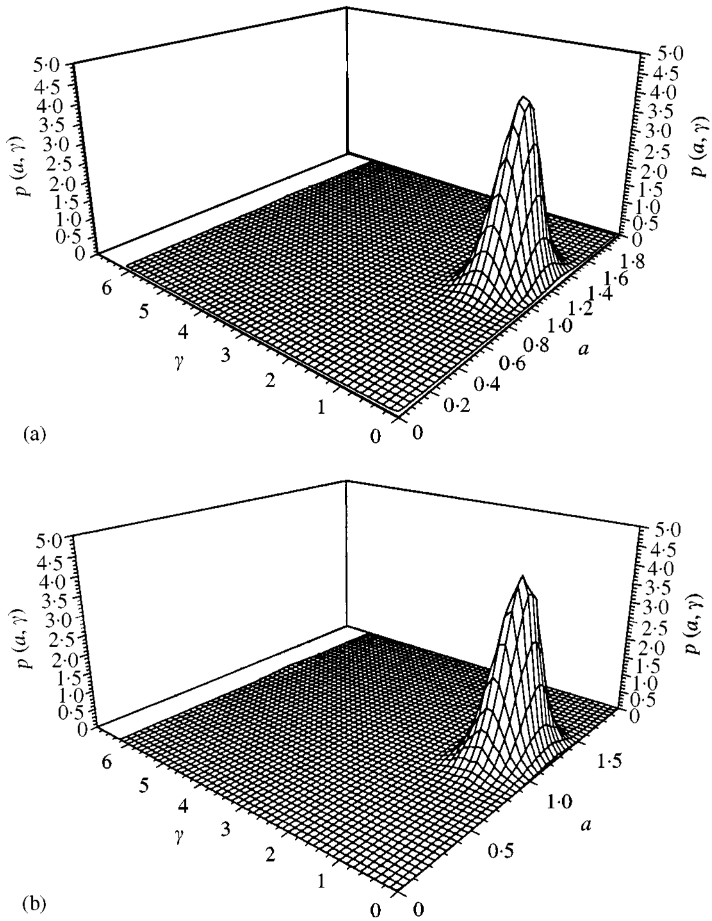


Figure 7. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}1$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

may occur. These two cases are also not within the frequency ratio interval of triple-valued amplitude, see curves E and C in Figure 3. So, the bifurcation of the stochastic jump of a Duffing oscillator under combined harmonic and white-noise excitations as the strength of non-linearity changes can also be roughly estimated based on the amplitude response curve of the same oscillator under pure harmonic excitation with different strength of non-linearity.

In the above examination of stochastic jump and its bifurcation, both harmonic and white-noise excitations are external. A stationary joint probability density of amplitude and phase of the Duffing oscillator under both external harmonic excitation and parametric white-noise excitation is shown in Figure 12, where the values of system parameters are the same as those in Figure 2 except white-noise excitation. It is seen from Figure 12 that both the densities from the stochastic averaging and path integration and from digital simulation are bimodal and thus jumps may occur in the system. It is also seen from the comparison between Figures 12(a) and 12(b) that the combination of the stochastic averaging and path integration still work well in this case.

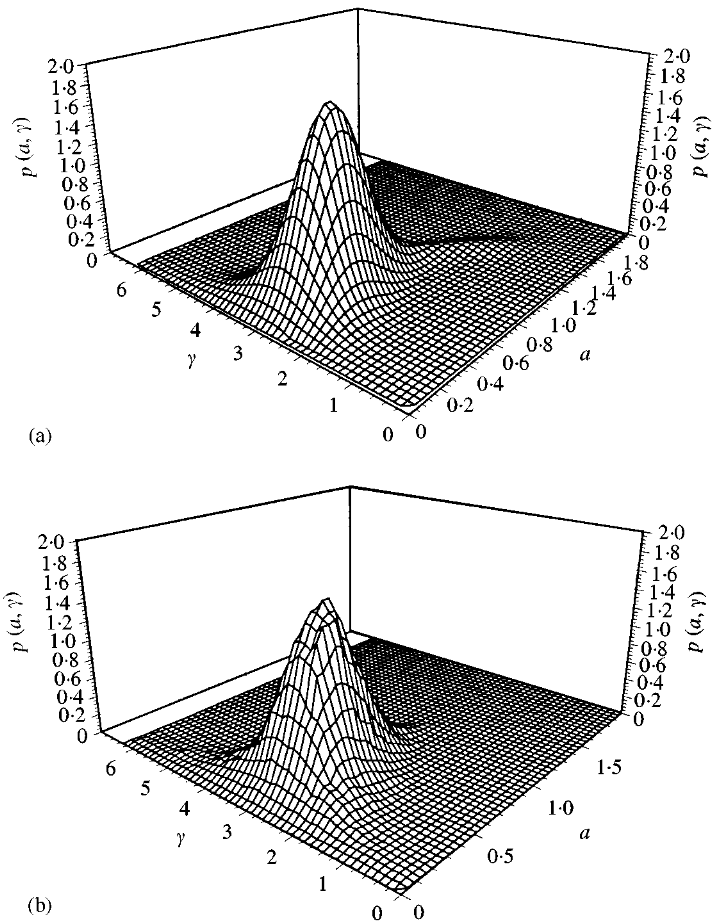
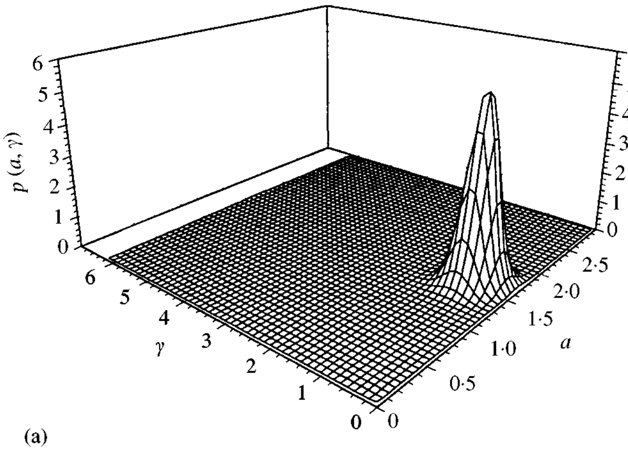


Figure 8. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}15$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

7. CONCLUDING REMARKS

In the present paper, a stochastic averaging procedure for strongly non-linear oscillators under external and (or) parametric excitations of both harmonic and white-noise forces has been developed by using the generalized harmonic functions. The procedure has been applied to a Duffing oscillator with hardening stiffness under combined externally harmonic excitation and externally and parametrically white-noise excitations. The averaged FPK equation has been solved numerically by using the technique of path integration and the results have been verified by those from the digital simulation of original equation. Based on the stationary joint probability densities of amplitude and phase obtained from the stochastic averaging and path integration and those from the digital simulation of original equation, the stochastic jump of a Duffing oscillator under combined harmonic and white-noise excitations and the bifurcation of the stochastic jump as the system's parameters change have been examined in detail. It has been shown that the



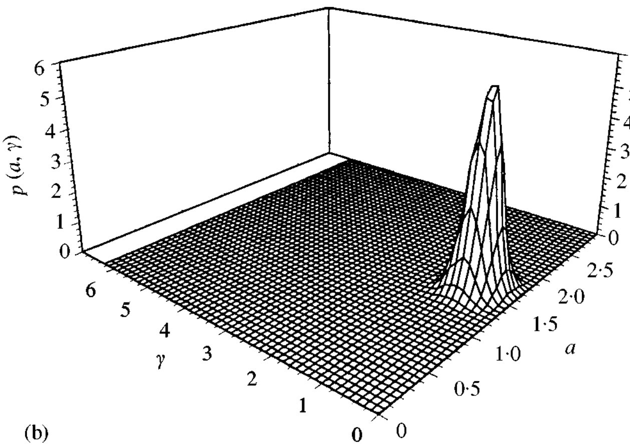


Figure 9. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}3$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

stochastic jump is essentially a transition from one more probable motion to another or *vice versa* and it is related to bimodal joint probability density. It has also been shown that the bifurcation of the stochastic jump as the frequency ratio, the amplitude of harmonic excitation and the strength of non-linearity change can be roughly estimated based on the amplitude response curves of the same oscillator with appropriate parameter values under pure harmonic excitation.

The proposed stochastic averaging method is applicable to the case of asymmetrical excitation and response and thus it may be applied to study the non-conventional stochastic resonance [42]. However, to obtain the concrete conclusion much work has to be done. The level of excitation intensity at which the proposed method is applicable depends on the system studied. For the system governed by equation (38), it is expected that the proposed method works well for the white-noise intensity less than 0·05.

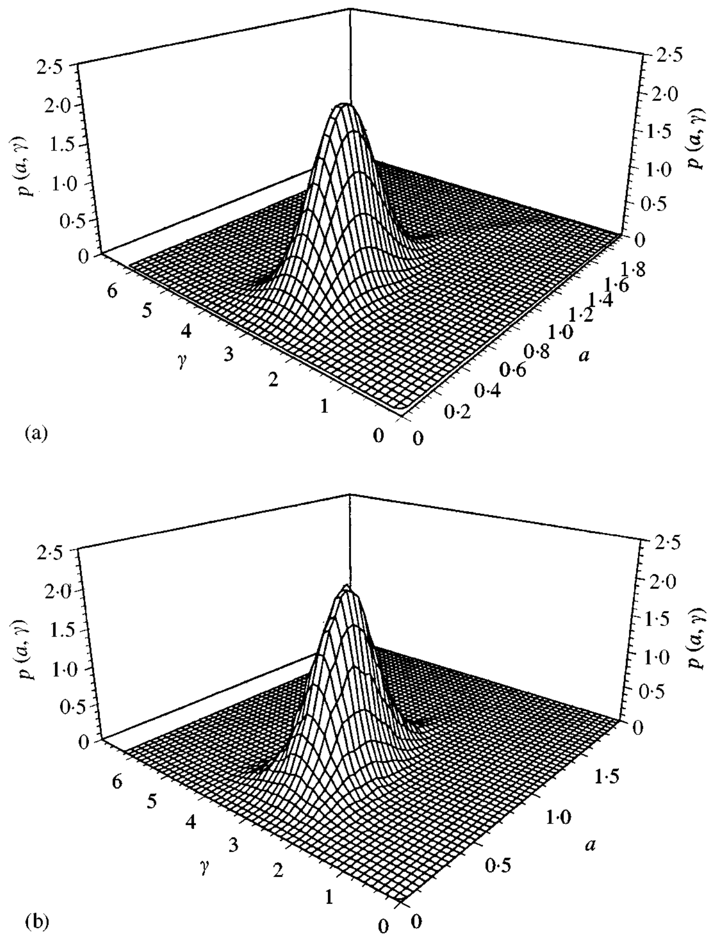


Figure 10. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}2$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

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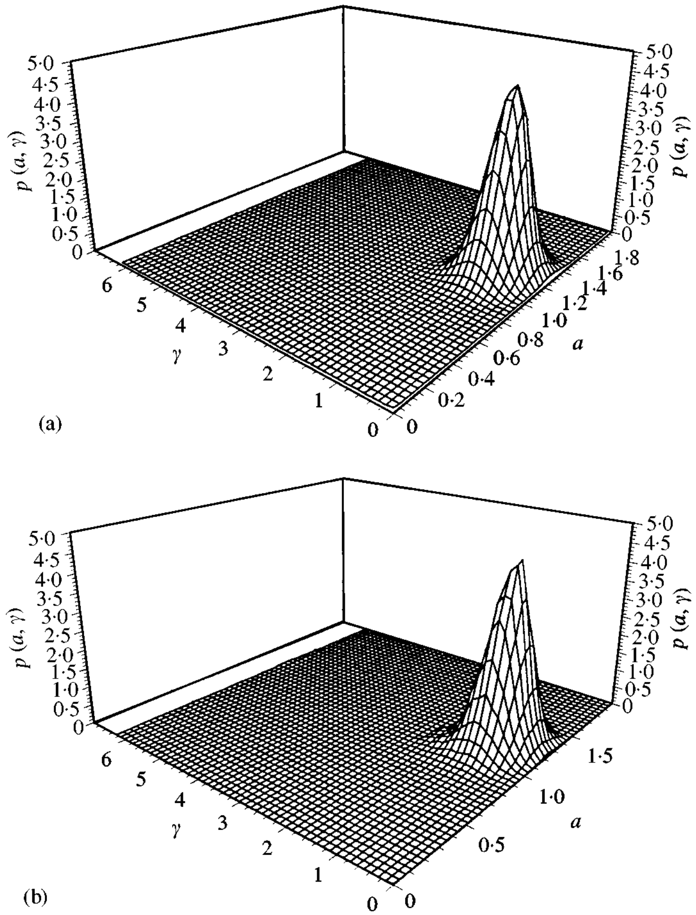

Figure 11. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}5$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}004$, $D_2 = 0{\cdot}0$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

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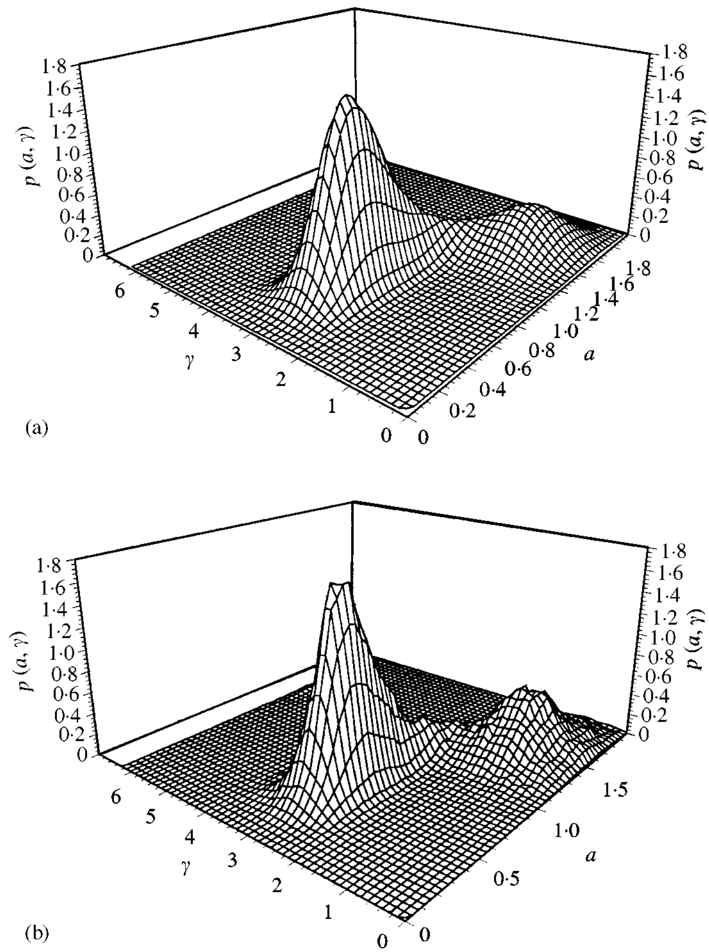

Figure 12. Stationary probability density $p(a, \gamma)$ of system (38). $\alpha = 0{\cdot}3$, $\beta = 0{\cdot}1$, $\omega = 1{\cdot}0$, $\Omega = 1{\cdot}2$, $E = 0{\cdot}2$, $D_1 = 0{\cdot}0$, $D_2 = 0{\cdot}02$ (a) by the stochastic averaging and path integration; (b) from the digital simulation of equation (38).

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